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1992 J. Phys. A: Math. Gen. 25 3039

(http://iopscience.iop.org/0305-4470/25/10/029)

J. Phys. A: Math. Gen. 25 (1992) 3039–3050. Printed in the UK

Fluctuations of Casimir forces on finite objects: II. Flat circular disk

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Received 24 October 1991

Abstract. The zero-point fluctuations of a scalar field with Neumann boundary conditions are investigated on a flat circular disk. The Helmholtz equationis is separated in oblate spheroidal coordinates; the mean-square force acting perpendicularly on the disk is calculated by picturing the disk as an oblate spheroid with zero eccentricity. The force is averaged over a time T, and the disk is taken small compared with cT. The results show that the fluctuations on a disk are roughly equal to those on a small sphere. For a one-sided disk divergences arise from the sharp edge, but the mean-square force has the same power dependence on T as for a hemisphere.

1. Introduction and outline

The present paper aims to extend the investigation of the fluctuations of Casimir forces to a finite, flat circular disk. As shown in the preceding paper (Eberlein 1992) the fluctuations can, with reasonable effort, be evaluated on the surfaces of spheres and hemispheres. It remains an open question to what extent the fluctuations are influenced by the geometrical shape of the object under consideration, e.g. one could ask whether or not a sphere may be regarded as a sufficiently good approximation also for a flat disk, as far as the fluctuations are concerned.

However, the mathematical handling of an object that is flat, in the strict sense of the word, turns out to be a quite demanding, but challenging task. A flat disk has an infinitely sharp rim which makes the mathematical boundary value problem for the wave or the Helmholtz equation ill-defined.An alternative way of tackling this problem is to start with an oblate spheroid and consider it in the limit of zero eccentricity which corresponds to a flat circular disk.

Since the mathematics required for treating the problem of a spheroid does certainly not belong to a physicist's standard repertoire , some mathematical tools concerning the Helmholtz equation in spheroidal coordinates will have to be accumulated first. Based on these foundations, subject of section 2, the fluctuations on a flat circular disk are evaluated in section 3; a two-sided disk is treated as a limit of a spheroid, and a one-sided disk as a limit of a hemispheroid. In both cases the disk is taken to have a radius small compared to the time T of time averaging, i.e. this paper deals exclusively with the long-wavelengths limit.

The basic way the calculation runs is as that for a small sphere or hemisphere, respectively, expounded in the preceding paper; and it is helpful to keep this in mind

0305-4470/92/103039+12$04.50

in order not to be suffocated by too much mathematical detail. For simplicity the calculations will be restricted to the scalar field with Neumann boundary conditions, since for spheres and hemispheres this was seen to be very close to the Maxwell field as regards the fluctuations.

2. The Helmholtz equation in oblate spheroidal coordinates

2.1. Separation of the Helmholtz equation

Spheroidal wave functions represent a fairly special topic within mathematical physics. An excellent, thorough survey of the subject in a clear presentation which allows one to immediately extract formulae for the use in physical problems, is given in the book by Flammer (1957, FL for short henceforth)†.

The oblate spheroidal coordinates are related to Cartesian coordinates via

$$\begin{aligned} x &= R\sqrt{(1-\eta^2)(\xi^2+1)}\ \cos\varphi \\ y &= R\sqrt{(1-\eta^2)(\xi^2+1)}\ \sin\varphi \\ z &= R\eta\xi \end{aligned} \tag{2.1}$$

with the z axis being the axis of revolution and the parameter space given by

$$-1 \leqslant \eta \leqslant 1 \qquad 0 \leqslant \xi < \infty \qquad 0 \leqslant \varphi \leqslant 2\pi .$$

An oblate ellipsoid of revolution is then uniquely described by an equation $\xi =$ constant; its semi-major axis is equal to $R\sqrt{\xi^2+1}$ and its semi-minor axis to $R\xi$. The degenerate case $\xi = 0$ corresponds to a flat circular disk of radius R; the other limiting case $\xi \to \infty$ leads back to spherical coordinates with $R\xi \to r$, $\eta \to \cos\theta$, and φ, of course, stays as it is.

The coordinates (η, ξ, φ) form a right-handed orthonormal system. The metric coefficients in

$$\mathrm{d}s^2 = h_\eta^2 \mathrm{d}\eta^2 + h_\xi^2 \mathrm{d}\xi^2 + h_\varphi^2 \mathrm{d}\varphi^2$$

read (FL 2.2.2b)

$$\begin{aligned} h_\eta &= R\sqrt{\frac{\xi^2+\eta^2}{1-\eta^2}} \\ h_\xi &= R\sqrt{\frac{\xi^2+\eta^2}{\xi^2+1}} \\ h_\varphi &= R\sqrt{(1-\eta^2)(\xi^2+1)}\ . \end{aligned} \tag{2.2}$$

With the abbreviation

$$c = kR \tag{2.3}$$

† It also contains an extensive list of references and two tables indicating how the notations of several other texts differ. The present author, however, sticks to Flammer's book in all questions of notation, normalization, etc.

the Helmholtz equation turns into (FL 2.2.3b)

$$\left[\frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial}{\partial\eta}+\frac{\partial}{\partial\xi}(\xi^2+1)\frac{\partial}{\partial\xi} + \frac{\xi^2+\eta^2}{(\xi^2+1)(1-\eta^2)}\frac{\partial^2}{\partial\varphi^2}+c^2(\xi^2+\eta^2)\right]\psi=0.$$

Separation of variables leads to a solution (FL 2.2.5b)†

$$\psi_{mn}=\mathcal{S}_{mn}(-\mathrm{i}c,\eta)\,\mathcal{R}_{mn}(-\mathrm{i}c,\mathrm{i}\xi)\,\mathrm{e}^{\mathrm{i}m\varphi}$$

with $\mathcal{S}$ and $\mathcal{R}$ satisfying the differential equations (FL 2.2.8, 9)

$$\frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial}{\partial\eta}\mathcal{S}_{mn}(-\mathrm{i}c,\eta)\right]+\left[\lambda_{mn}+c^2\eta^2-\frac{m^2}{1-\eta^2}\right]\mathcal{S}_{mn}(-\mathrm{i}c,\eta)=0 \qquad (2.4)$$

$$\frac{\partial}{\partial\xi}\left[(\xi^2+1)\frac{\partial}{\partial\xi}\mathcal{R}_{mn}(-\mathrm{i}c,\mathrm{i}\xi)\right]-\left[\lambda_{mn}-c^2\xi^2-\frac{m^2}{1-\eta^2}\right]\mathcal{R}_{mn}(-\mathrm{i}c,\mathrm{i}\xi)=0. \qquad (2.5)$$

In imitation of the spherical limit the functions $\mathcal{S}_{mn}(-\mathrm{i}c,\eta)$ are henceforward referred to as the *angle* functions, and the functions $\mathcal{R}_{mn}(-\mathrm{i}c,\mathrm{i}\xi)$ as the *radial* functions.

2.2. *The angle functions*

For $c=0$ the differential equation (2.4) is satisfied by the associated Legendre functions. The angle functions of the first kind (i.e. those finite at $\eta=\pm1$) therefore reduce to the associated Legendre functions of the first kind $P_n^m(\eta)$ in the limit $c\to0$. It is appropriate to expand (FL 3.1.3b)

$$\mathcal{S}_{mn}(-\mathrm{i}c,\eta)={\sum_{r=0,1}^{\infty}}{}' d_r^{mn}(-\mathrm{i}c)P_{m+r}^m(\eta) \qquad (2.6)$$

where the prime indicates a restricted summation over even or odd r if $n-m$ is even or odd, respectively. The coefficients in the above expansion may be obtained from the following recursion relation (FL 3.1.4)

$$\frac{(2m+r+2)(2m+r+1)c^2}{(2m+2r+3)(2m+2r+5)}d_{r+2}^{mn}(c)+\Big[(m+r)(m+r+1)-\lambda_{mn}(c) + \frac{2(m+r)(m+r+1)-2m^2-1}{(2m+2r-1)(2m+2r+3)}c^2\Big]d_r^{mn}(c) + \frac{r(r-1)c^2}{(2m+2r-3)(2m+2r-1)}d_{r-2}^{mn}(c)=0 \qquad (r\geqslant0) \qquad (2.7)$$

† This solution is generated from the one in prolate spheroidal coordinates by the transformation $\xi\to\mathrm{i}\xi$, $c\to-\mathrm{i}c$. The notation that originates from this transformation is kept here for transparency of reference.

with the eigenvalue λ_{mn} given by the expansion (FL 3.1.17–19)

$$\lambda_{mn}(c) = \sum_{k=0}^{\infty} \ell_{2k}^{mn} c^{2k}$$
$$\ell_0^{mn} = n(n+1)$$
$$\ell_2^{mn} = \frac{1}{2}\left[1 - \frac{(2m-1)(2m+1)}{(2n-1)(2n+3)}\right].$$

The analogy to the small sphere (see preceding paper) where Watson's Lemma was applied suggests that one truncates the expansion for the coefficients d_r^{mn} from a certain order of powers of c on, since small values of c deliver the important contributions to the fluctuations in the end. As $d_r^{mn} \propto c^{\pm(r-n+m)}$ for $n - m \lessgtr r$, there is only a finite number of ds to be included if one restricts oneself to a certain order of c. Up to order c^2 the coefficients d_r^{mn} are given in appendix A for low values of n and m.

2.3. The radial functions

The solutions $\mathcal{R}_{mn}^{(1)}(-\mathrm{i}c, \mathrm{i}\xi)$ and $\mathcal{R}_{mn}^{(2)}(-\mathrm{i}c, \mathrm{i}\xi)$ of the differential equation (2.5) reduce to the spherical Bessel functions $j_n(c\xi)$ and $y_n(c\xi)$, respectively, in the limit of $c \to 0$. Similarly to the expansion for the angle functions (2.6) the functions $\mathcal{R}_{mn}^{(1)}$ and $\mathcal{R}_{mn}^{(2)}$ can be expanded in terms of $j_{m+r}(c\xi)$ and $y_{m+r}(c\xi)$, respectively, where again the coefficients d_r^{mn} are involved.

As it turns out, the angle and the radial functions of the same kind are proportional to each other, which can be used to expand the radial functions of the second kind $\mathcal{R}_{mn}^{(2)}$ in terms of the regular and irregular associated Legendre functions, P_ℓ^m and Q_ℓ^m. This type of expansion is then appropriate for putting $\xi = 0$. For $\mathcal{R}_{mn}^{(1)}$ the expansion in terms of the spherical Bessel functions $j_{m+r}(c\xi)$ allows one to determine the value at $\xi = 0$. Skipping all intermediate technicalities† one can write down for the derivatives of $\mathcal{R}_{mn}^{(1)}$ and $\mathcal{R}_{mn}^{(2)}$ with respect to ξ at $\xi = 0$

$$\mathcal{R}_{mn}^{(1)\prime}(-\mathrm{i}c, \mathrm{i}0)$$
$$= 0 \qquad \text{for } (n-m) \text{ even} \qquad (2.8)$$
$$= \frac{\mathrm{i}^{n-m-1} 2^{m+1} m! c^{m+1} d_1^{mn}(-\mathrm{i}c)}{(2m+1)(2m+3)\sum_{r=1}^{\infty}{}' d_r^{mn}(-\mathrm{i}c)\frac{(2m+r)!}{r!}} \qquad \text{for } (n-m) \text{ odd} \qquad (2.9)$$
$$\mathcal{R}_{mn}^{(2)\prime}(-\mathrm{i}c, \mathrm{i}0)$$
$$= \frac{\mathrm{i}^{n-m}(2m+1)\sum_{r=0}^{\infty}{}' d_r^{mn}(-\mathrm{i}c)\frac{(2m+r)!}{r!}}{2^m m! c^{m+1} d_0^{mn}(-\mathrm{i}c)} \qquad \text{for } (n-m) \text{ even} \qquad (2.10)$$
$$= \frac{\mathrm{i}^{n-m-1}(2m-3)(2m-1)m! c^{m-2}\pi}{2^{2n-m+1}(2m)! d_{-2m+1}^{mn}(-\mathrm{i}c)\sum_{r=1}^{\infty}{}' d_r^{mn}(-\mathrm{i}c)\frac{(2m+r)!}{r!}}$$
$$\times \left[\frac{(n+m+1)!}{\left(\frac{n-m+1}{2}\right)!\left(\frac{n+m+1}{2}\right)!}\right]^2 \qquad \text{for } (n-m) \text{ odd} \qquad (2.11)$$

† For a detailed discussion of all this see FL, chapter 4.

which are the quantities that gain importance in the following. Note especially

$$\mathcal{R}^{(2)\prime}_{mn}(-\mathrm{i}c, \mathrm{i}0) \propto c^{-n-1} \tag{2.12}$$

$$\mathcal{R}^{(1)\prime}_{mn}(-\mathrm{i}c, \mathrm{i}0) \propto c^{n} \qquad \text{for } (n-m) \text{ odd.} \tag{2.13}$$

3. Fluctuations on an oblate spheroid †

3.1. Integration of the stress over the spheroid

Let the spheroid be given by $\xi = \xi_0 = \mathrm{constant}$. The Neumann boundary condition which requires the normal derivative of the scalar field ψ to vanish on the surface, i.e.

$$\nabla_\xi \psi = 0 \tag{3.1}$$

reduces the relevant stress-tensor components to

$$S_{\xi\xi} = \frac{1}{2}\left(\frac{\partial\psi}{\partial t}\right)^2 - \tfrac{1}{2}\left(\nabla_\varphi\psi\right)^2 - \tfrac{1}{2}\left(\nabla_\eta\psi\right)^2$$

$$S_{\varphi\xi} = S_{\eta\xi} = 0.$$

The infinitesimal surface element on the spheroid is

$$\mathrm{d}\boldsymbol{\Sigma} = h_\varphi h_\eta \mathrm{d}\varphi \mathrm{d}\eta \hat{e}_\xi = R^2\sqrt{(\xi_0^2+\eta^2)(\xi_0^2+1)}\mathrm{d}\varphi\mathrm{d}\eta\hat{e}_\xi$$

with $\hat{e}_\xi$ denoting the unit vector in the ξ direction. From $\mathrm{d}F_i = S_{ij}\mathrm{d}\Sigma_j$ it follows that the force is perpendicular to the surface of the spheroid, i.e.

$$\mathrm{d}\boldsymbol{F} = S_{\xi\xi}\mathrm{d}\boldsymbol{\Sigma}\,.$$

Since

$$\hat{e}_\xi \cdot \hat{e}_z = \sqrt{\frac{\xi_0^2+1}{\xi_0^2+\eta^2}}\eta$$

and

$$\nabla_\varphi\psi = \frac{1}{h_\varphi}\frac{\partial\psi}{\partial\varphi} \qquad \nabla_\eta\psi = \frac{1}{h_\eta}\frac{\partial\psi}{\partial\eta}$$

the z component (i.e. the one along the axis of rotational symmetry) of the force acting on the spheroid reads

$$\begin{aligned} F_z &= \int_0^{2\pi}\mathrm{d}\varphi\int_{-1}^{1}\mathrm{d}\eta\, R^2(\xi_0^2+1)\eta\, S_{\xi\xi} \\ &= \int_0^{2\pi}\mathrm{d}\varphi\int_{-1}^{1}\mathrm{d}\eta\,\frac{\eta}{2}\left[R^2(\xi_0^2+1)\left(\frac{\partial\psi}{\partial t}\right)^2 - \frac{1}{1-\eta^2}\left(\frac{\partial\psi}{\partial\varphi}\right)^2\right. \\ &\qquad \left. -\frac{(1-\eta^2)(\xi_0^2+1)}{\xi_0^2+\eta^2}\left(\frac{\partial\psi}{\partial\eta}\right)^2\right]. \end{aligned} \tag{3.2}$$

† The basic reasoning of this section is analogous to the one for the sphere spelled out in the preceding paper, especially in section 3. To save space the general outline will not be repeated here, and the reader is expected to be familiar with the comparatively simple spherical case.

3.2. The normal modes

Defining the 'spheroidal harmonics'

$$\mathcal{Y}_\ell^m(c,\eta) = (-1)^m \sqrt{\frac{(\ell - |m|)!}{\ell + |m|)!}} \sqrt{\frac{2\ell+1}{4\pi}}\, \mathcal{S}_{|m|\ell}(-\mathrm{i}c,\eta)\,\mathrm{e}^{\mathrm{i}m\varphi}$$

one can write down the normal modes for the scalar field,

$$\psi = \frac{4\pi}{(2\pi)^{3/2}} \sum_{\ell,m} \frac{\mathrm{e}^{-\mathrm{i}\delta_{m\ell}}\mathrm{i}^\ell}{\sqrt{2k}} \left[\cos\delta_{m\ell}\mathcal{R}^{(1)}_{m\ell}(-\mathrm{i}c,\mathrm{i}\xi) + \sin\delta_{m\ell}\mathcal{R}^{(2)}_{m\ell}(-\mathrm{i}c,\mathrm{i}\xi)\right] \times \mathcal{Y}_\ell^{m*}(\hat{k})\mathcal{Y}_\ell^m(c,\eta)$$

in close analogy to spherical coordinates. The boundary condition (3.1) fixes

$$\tan\delta_{m\ell} = -\frac{\mathcal{R}^{(1)\prime}_{m\ell}(-\mathrm{i}c,\mathrm{i}\xi_0)}{\mathcal{R}^{(2)\prime}_{m\ell}(-\mathrm{i}c,\mathrm{i}\xi_0)}\,. \tag{3.3}$$

The relevant matrix element of the force (3.2) on a spheroid is then, Lorentzian time averaging already included,

$$\begin{aligned}\langle k,k' \mid \overline{F_z} \mid 0\rangle = \int_0^{2\pi}\mathrm{d}\varphi\int_{-1}^{1}\mathrm{d}\eta\,\frac{1}{\pi}\sum_{\ell,m}\sum_{\ell',m'} \frac{\mathrm{e}^{\mathrm{i}\delta_{m\ell}+\mathrm{i}\delta_{m'\ell'}}(-\mathrm{i})^{\ell+\ell'}}{\sqrt{kk'}}\,\mathrm{e}^{-(\omega+\omega')T}\,\eta \\ \times\Bigg[-cc'(\xi_0^2+1)\mathcal{Y}_\ell^{m*}(c,\eta)\mathcal{Y}_{\ell'}^{m'*}(c',\eta) + \frac{mm'}{1-\eta^2}\mathcal{Y}_\ell^{m*}(c,\eta)\mathcal{Y}_{\ell'}^{m'*}(c',\eta) \\ -\frac{(1-\eta^2)(\xi_0^2+1)}{\xi_0^2+\eta^2}\frac{\partial\mathcal{Y}_\ell^{m*}(c,\eta)}{\partial\eta}\frac{\partial\mathcal{Y}_{\ell'}^{m'*}(c',\eta)}{\partial\eta}\Bigg] \\ \times\,\mathcal{Y}_\ell^m(\hat{k})\mathcal{Y}_{\ell'}^{m'}(\hat{k}')\left[\cos\delta_{m\ell}\mathcal{R}^{(1)}_{m\ell}(-\mathrm{i}c,\mathrm{i}\xi_0) + \sin\delta_{m\ell}\mathcal{R}^{(2)}_{m\ell}(-\mathrm{i}c,\mathrm{i}\xi_0)\right] \\ \times\left[\cos\delta_{m'\ell'}\mathcal{R}^{(1)}_{m'\ell'}(-\mathrm{i}c',\mathrm{i}\xi_0) + \sin\delta_{m'\ell'}\mathcal{R}^{(2)}_{m'\ell'}(-\mathrm{i}c',\mathrm{i}\xi_0)\right].\end{aligned} \tag{3.4}$$

For a hemispheroid the above expression applies almost unaltered; only the η integration runs from 0 to 1 rather than from -1 to 1.

3.3. The spheroid

The integration over the spheroid involves the three integrals

$$\mathcal{J}_1 = \int_0^{2\pi}\mathrm{d}\varphi\int_{-1}^{1}\mathrm{d}\eta\,\eta\,\mathcal{Y}_\ell^{m*}(c,\eta)\mathcal{Y}_{\ell'}^{m'*}(c',\eta) \tag{3.5}$$

$$\mathcal{J}_2 = \int_0^{2\pi}\mathrm{d}\varphi\int_{-1}^{1}\mathrm{d}\eta\,\eta\,\frac{-mm'}{1-\eta^2}\mathcal{Y}_\ell^{m*}(c,\eta)\mathcal{Y}_{\ell'}^{m'*}(c',\eta) \tag{3.6}$$

$$\mathcal{J}_3 = \int_0^{2\pi}\mathrm{d}\varphi\int_{-1}^{1}\mathrm{d}\eta\,\eta\,\frac{1-\eta^2}{\xi_0^2+\eta^2}\frac{\partial\mathcal{Y}_\ell^{m*}(c,\eta)}{\partial\eta}\frac{\partial\mathcal{Y}_{\ell'}^{m'*}(c',\eta)}{\partial\eta}\,. \tag{3.7}$$

One easily recognizes that all three integrals are proportional to $\delta_{m\,-m'}$. Furthermore, the integrals vanish for $|\ell-\ell'|$ even, as the expansion (2.6) for $\mathcal{S}_{m\ell}$ proceeds in steps of 2 in the lower index of the Legendre polynomials. The non-vanishing results of the above integrals are listed in appendix B.1 for the lowest ℓ and ℓ'.

The next step after the surface integration is the evaluation of

$$\Delta\overline{F_z}^2 = \frac{1}{2}\int \mathrm{d}^3\boldsymbol{k}\int \mathrm{d}^3\boldsymbol{k}'\ \left|\langle \boldsymbol{k},\boldsymbol{k}' \mid \overline{F_z}\mid 0\rangle\right|^2 . \tag{3.8}$$

As the functions $\mathcal{S}_{m\ell}$ are solutions of the differential equation (2.4), general theorems about the Sturm–Liouville problem (see Morse and Feshbach 1953, pp 726–729) ensure that the $\mathcal{Y}$ functions form a complete orthogonal set of eigenfunctions on the interval [-1,1]. Hence it is

$$\int \mathrm{d}^3\boldsymbol{k}\, \mathcal{Y}_\ell^{m*}(\hat{\boldsymbol{k}})\mathcal{Y}_{\ell'}^{m'}(\hat{\boldsymbol{k}}) = \delta_{mm'}\delta_{\ell\ell'}\int_0^\infty \mathrm{d}k\, k^2 .$$

If one abbreviates

$$\mathcal{G}_\ell^m(c,\xi_0) \equiv \cos\delta_{m\ell}\mathcal{R}_{m\ell}^{(1)}(-\mathrm{i}c,\mathrm{i}\xi_0) + \sin\delta_{m\ell}\mathcal{R}_{m\ell}^{(2)}(-\mathrm{i}c,\mathrm{i}\xi_0) \tag{3.9}$$

(3.8) yields

$$\begin{aligned}\Delta\overline{F_z}^2 = \frac{1}{2\pi^2}&\int_0^\infty \mathrm{d}k\int_0^\infty \mathrm{d}k'\, kk'\,\mathrm{e}^{-2(k+k')T}\\ &\times\left[\left|\mathcal{G}_0^0(c,\xi_0)\right|^2\left|\mathcal{G}_1^0(c',\xi_0)\right|^2(\xi_0^2+1)^2\left(cc'\frac{\sqrt{3}}{3}+\frac{2\sqrt{3}}{9}c^2\right)^2\right.\\ &+\left|\mathcal{G}_1^0(c,\xi_0)\right|^2\left|\mathcal{G}_0^0(c',\xi_0)\right|^2(\xi_0^2+1)^2\left(cc'\frac{\sqrt{3}}{3}+\frac{2\sqrt{3}}{9}c'^2\right)^2\\ &+\left|\mathcal{G}_1^0(c,\xi_0)\right|^2\left|\mathcal{G}_2^0(c',\xi_0)\right|^2(\xi_0^2+1)^2\left(cc'\frac{2\sqrt{15}}{15}+2\sqrt{15}\right)^2\\ &+\left|\mathcal{G}_2^0(c,\xi_0)\right|^2\left|\mathcal{G}_1^0(c',\xi_0)\right|^2(\xi_0^2+1)^2\left(cc'\frac{2\sqrt{15}}{15}+2\sqrt{15}\right)^2\\ &+2\left|\mathcal{G}_1^1(c,\xi_0)\right|^2\left|\mathcal{G}_2^1(c',\xi_0)\right|^2\left(cc'(\xi_0^2+1)\frac{\sqrt{5}}{5}-(\xi_0^2+1)\frac{\sqrt{5}}{2}+\frac{\sqrt{5}}{2}\right)^2\\ &+2\left|\mathcal{G}_2^1(c,\xi_0)\right|^2\left|\mathcal{G}_1^1(c',\xi_0)\right|^2\left(cc'(\xi_0^2+1)\frac{\sqrt{5}}{5}-(\xi_0^2+1)\frac{\sqrt{5}}{2}+\frac{\sqrt{5}}{2}\right)^2\\ &\left.+\cdots\right].\end{aligned}$$

It remains to calculate the $|\mathcal{G}_\ell^m|^2$ in the limit $\xi_0 = 0$. From (3.9) and (3.3) it follows that

$$|\mathcal{G}_\ell^m(c,\xi_0)|^2 = \frac{1}{c^2(\xi_0^2+1)^2}\frac{1}{\left[\mathcal{R}_{m\ell}^{(1)\prime}(-\mathrm{i}c,\mathrm{i}\xi_0)\right]^2+\left[\mathcal{R}_{m\ell}^{(2)\prime}(-\mathrm{i}c,\mathrm{i}\xi_0)\right]^2} \tag{3.10}$$

where the Wronskian of $\mathcal{R}_{m\ell}^{(1)}$ and $\mathcal{R}_{m\ell}^{(2)}$ (FL 4.1.21)

$$\mathcal{R}_{m\ell}^{(1)}(-\mathrm{i}c,\mathrm{i}\xi_0)\mathcal{R}_{m\ell}^{(2)\prime}(-\mathrm{i}c,\mathrm{i}\xi_0) - \mathcal{R}_{m\ell}^{(1)\prime}(-\mathrm{i}c,\mathrm{i}\xi_0)\mathcal{R}_{m\ell}^{(2)}(-\mathrm{i}c,\mathrm{i}\xi_0) = \frac{1}{c(\xi_0^2+1)}$$

was employed. With a quick look at (2.12), (2.13) and (2.8) one convinces oneself that for small c (3.10) is governed by the functions $\mathcal{R}_{m\ell}^{(2)\prime}(-\mathrm{i}c,\mathrm{i}0)$. Equation (2.10) yields

$$\mathcal{R}_{00}^{(2)\prime}(-\mathrm{i}c,\mathrm{i}0) = \frac{1}{c} + \mathrm{O}(c) \tag{3.11}$$

$$\mathcal{R}_{11}^{(2)\prime}(-\mathrm{i}c,\mathrm{i}0) = \frac{3}{c^2} + \mathrm{O}(1) \tag{3.12}$$

$$\mathcal{R}_{02}^{(2)\prime}(-\mathrm{i}c,\mathrm{i}0) = \frac{45}{c^3} + \mathrm{O}\left(\frac{1}{c^2}\right) . \tag{3.13}$$

From (2.11) one easily obtains

$$\mathcal{R}_{01}^{(2)\prime}(-\mathrm{i}c,\mathrm{i}0) = \frac{3\pi}{2c^2} + \mathrm{O}(1). \tag{3.14}$$

In order to get $\mathcal{R}_{12}^{(2)\prime}$ one needs to determine the expansion coefficient d_{-1}^{12} first. Noting that $d_r^{mn} = 0$ for $r \leqslant -2m$ (FL 3.5.6) the leading term is found from the recursion formula (2.7)

$$d_{-1}^{12}(-\mathrm{i}c) = \frac{c^2}{15} .$$

Then (2.11) entails

$$\mathcal{R}_{12}^{(2)\prime}(-\mathrm{i}c,\mathrm{i}0) = \frac{45\pi}{4c^3} . \tag{3.15}$$

After insertion of (3.11) to (3.15) into (3.10) and the expression for $\Delta\overline{F_z}^2$ the remaining integrations over k and k', i.e. over c and c', are trivial, and eventually one finds for the fluctuations on a flat circular disk, in leading order

$$\Delta\overline{F_z}^2 \sim \frac{13R^2}{3\times 2^4\pi^6 T^{10}}\left(\pi R^2\right)^2 \tag{3.16}$$

or, expressed numerically,

$$\frac{\Delta\overline{F_z}^2}{\left(\pi R^2\right)^2} \sim 9.0\times 10^{-5}\frac{\pi R^2}{T^{10}} . \tag{3.17}$$

3.4. *The hemispheroid*

Having evaluated the fluctuations on a spheroid one would assume that there will not be any great difficulties in doing the same calculation for the hemispheroid in order to estimate the fluctuations on a one-sided disk, i.e. a disk where the correlation between the two sides are artificially switched off. All one would have to alter is the integration range for η in (3.4) to run from 0 rather than from -1. However, one thing that was not mentioned so far is that for $\xi_0 = 0$ certain modes of $\nabla_\eta \psi$ have a singularity at the edge of the disk described by $\eta = 0$. More precisely, the modes behave like

$$\nabla_\eta \psi \propto \frac{1}{\eta} \propto \frac{1}{\sqrt{s}} \tag{3.18}$$

near the edge $\eta = 0$ if s denotes the distance of a point on the disk from the edge†.

For the whole spheroid one did not need to bother about these singularities since they appear as $1/\eta$ under the integral $\int_{-1}^{1} \mathrm{d}\eta$. Since the forces on the two sides cancel at the edge, the Cauchy principal value is the appropriate prescription to get physically sensible results from those integrals.

Now, in contrast, there is no getting away from these singularities. It was Bouwkamp (1946) who first examined the singularities along the sharp rim in Sommerfeld's diffraction problem of the half-plane. He found exactly the same behaviour of the singularity as indicated in (3.18), arising of course from the idealized concept of an infinitely sharp edge. More generally, at a sharp wedge with the opening angle α the singularities go like $s^{\nu-1}$ where $\nu = \pi/(2\pi - \alpha)$. Van Bladel (1991) shows this in a quick, but comprehensible way. Furthermore, as he also points out, the singular behaviour of the field is limited to the region 'near' the edge where s is much smaller than the wavelength. This is, however, irrelevant in the present case since it is the long-wavelength limit that is under consideration here. Nevertheless, despite the singularities in the field the energy density stays integrable with respect to *volume* integration, so that there is nothing wrong in the physics even in the idealized case of an infinitely sharp rim.

In the evaluation of the fluctuations on a one-sided disk the singularity enters due to the integral analogous to $\mathcal{J}_3$ in (3.7) over the hemispheroid. Appendix B.2 lists the non-vanishing results of all three integrals for low ℓ and ℓ'.

Keeping only lowest powers in c and c' and neglecting all terms that vanish for $\xi_0 = 0$, one arrives, along the same lines as before, at

$$\Delta \overline{F_z}^2 = \frac{1}{2\pi^2} \int_0^\infty \mathrm{d}k \int_0^\infty \mathrm{d}k'\, kk'\, \mathrm{e}^{-2(k+k')T} \bigg[\left|\mathcal{G}_0^0(c,\xi_0)\right|^2 \left|\mathcal{G}_0^0(c',\xi_0)\right|^2 \left(\frac{cc'}{4}\right)^2 + \left|\mathcal{G}_1^0(c,\xi_0)\right|^2 \left|\mathcal{G}_1^0(c',\xi_0)\right|^2 \left(\tfrac{3}{4} + \tfrac{3}{2}\ln\xi_0\right)^2 + 2\left|\mathcal{G}_1^1(c,\xi_0)\right|^2 \left|\mathcal{G}_1^1(c',\xi_0)\right|^2 \left(\tfrac{3}{4}\right)^2 + \cdots \bigg].$$

By use of (3.10) and (3.11) to (3.14) the final result for the fluctuations on a one-sided circular disk is then found to be in leading order

$$\Delta \overline{F_z}^2 \sim \frac{11}{2^{11}\pi^4 T^8} \left[1 + \frac{16}{11\pi^4}\left(1 + 2\ln\xi_0\right)^2\right] \left(\pi R^2\right)^2 \tag{3.19}$$

† Note that $s = R\eta^2/2$ for points that lie on the disk.

Since contributions of order $\ln \xi_0 R^2/T^{10}$ are not included here one has to impose the requirement

$$\left|\left(\frac{R}{T}\right)^2 \ln \xi_0\right| \ll 1 . \tag{3.20}$$

If one recalls the lengths of major and minor axes of the spheroid from section 2.1, the eccentricity ξ_0 readily turns out to be the quotient of thickness and diameter of the disk. Even if one could get down to disks with $\xi_0 \sim 10^{-5}$ there is still no difficulty in obeying (3.20), so that the result (3.19) is really the leading order contribution to the fluctuations on a one-sided disk. Putting $\xi_0 = 10^{-4}$ one obtains the numerical value

$$\frac{\Delta \overline{F_z}^2}{(\pi R^2)^2} \sim 3.1 \times 10^{-4} \frac{1}{T^8} . \tag{3.21}$$

4. Comparison and conclusions

The two main results of this paper, the mean-square-deviations of the forces acting perpendicularly on a two- or one-sided flat circular disk, given by equations (3.16) and (3.19), are to be compared with the results for the fluctuations of the scalar field with Neumann boundary conditions on a sphere, a hemisphere and an embedded piston (see preceding paper, table 2, upper row). All objects are taken to be small compared with the typical time T of time averaging†.

Comparison with sphere and hemisphere shows that the actual geometrical shape is indeed of minor importance; the two- and one-sided disks, respectively, yield the fluctuations with the same powers of R/T as their curved counterparts‡.

While, surprisingly, for the two-sided disk no difficulties occur in evaluating the fluctuations, the analogous calculation for the one-sided disk explicitly faces the singularities on the sharp rim. This reminds one of the unphysical idealization made for the one-sided disk where correlations between the two sides are removed; in reality there are, of course, very strong correlations between the two sides near the rim.

The main goal of this paper, however, was the comparison of the fluctuations on a (whole) disk (3.16) and on a sphere, leading to the conclusion that, as regards the fluctuations, any flat disk is reasonably well approximated by a sphere. In tackling more difficult problems than the fluctuations on a stationary, perfectly rigid object, like non-ideal boundary conditions or movable objects, one could consider a sphere rather than a disk, limiting the mathematical difficulties to a bearable extent and having a clear conscience about not doing too much wrong.

† For a large disk, i.e. where the correlation length is much smaller than geometrical extent of the disk, the fluctuations on the two sides are not correlated anyway, so that it makes no difference whether one considers an isolated disk or an embedded piston.

‡ This stays true also for the scalar field with Dirichlet boundary conditions as may be checked in a short calculation.

Appendix A. Expansion coefficients d_r^{mn} for low m, n

From the recursion formula (2.7) one finds†

$$d_0^{00}(-\mathrm{i}c) = 1 + \frac{c^2}{18} \qquad d_2^{00}(-\mathrm{i}c) = \frac{c^2}{9}$$

$$d_1^{01}(-\mathrm{i}c) = 1 + \frac{3c^2}{50} \qquad d_3^{01}(-\mathrm{i}c) = \frac{c^2}{25}$$

$$d_0^{11}(-\mathrm{i}c) = 1 + \frac{c^2}{50} \qquad d_2^{11}(-\mathrm{i}c) = \frac{c^2}{75}$$

$$d_0^{02}(-\mathrm{i}c) = -\frac{c^2}{45} \qquad d_2^{02}(-\mathrm{i}c) = 1 - \frac{23c^2}{882} \qquad d_4^{02}(-\mathrm{i}c) = \frac{6c^2}{245}$$

$$d_1^{12}(-\mathrm{i}c) = 1 + \frac{3c^2}{98} \qquad d_3^{12}(-\mathrm{i}c) = \frac{3c^2}{245}$$

where all terms of orders higher than c^2 are omitted.

The *d*s are normalized so that the angle functions (2.6) reduce exactly to the corresponding associated Legendre polynomial for $c = 0$. This is ensured here by requiring identical behaviour of the two functions at $\eta = 0$ according to

$$\begin{aligned} \mathcal{S}_{mn}(c,0) &= P_n^m(0) \qquad \text{for } (n-m) \text{ even,} \\ \mathcal{S}'_{mn}(c,0) &= P_n^{m\prime}(0) \qquad \text{for } (n-m) \text{ odd.} \end{aligned}$$

Appendix B. Surface integrations

B1. Spheroid

In the notation $\mathcal{J}_i(\ell, m; \ell')$ and with $\xi_0 = 0$ for the flat disk, one has

$$\mathcal{J}_1(0,0;1) = \frac{\sqrt{3}}{3} + \mathrm{O}(c^2, c'^2) \qquad \mathcal{J}_3(0,0;1) = \frac{2\sqrt{3}}{9}c^2 + \mathrm{O}(c^2c'^2, c^4, c'^4)$$

$$\mathcal{J}_1(1,0;2) = \frac{2\sqrt{15}}{15} + \mathrm{O}(c^2, c'^2) \qquad \mathcal{J}_3(1,0;2) = 2\sqrt{15} + \mathrm{O}(c^2, c'^2)$$

$$\mathcal{J}_1(1,1;2) = \frac{\sqrt{5}}{5} + \mathrm{O}(c^2, c'^2) \qquad \mathcal{J}_2(1,1;2) = \frac{\sqrt{5}}{2} + \mathrm{O}(c^2, c'^2)$$

$$\mathcal{J}_3(1,1;2) = -\frac{\sqrt{5}}{2} + \mathrm{O}(c^2, c'^2).$$

† A method for obtaining the *d*s from (2.7) is explained by Morse and Feshbach (1953) p. 1504. One has, however, to be careful in transferring results since their normalization is different from the one employed here. Besides, the recursion formula has a misprinted m^2 instead of $2m^2$ in one of the numerators.

B2. Hemispheroid

The integrals $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$ (see (3.5) to (3.7)) are now evaluated for the range $\int_0^1 d\eta$. All terms that vanish in the limit $\xi_0 \to 0$ are omitted, and so are higher terms in c and c' even if they are infinite for $\xi_0 \to 0$. Note that all three integrals are still proportional to $\delta_{m-m'}$.

$$\mathcal{J}_1(0,0;0) = \frac{1}{4} + O(c^2, c'^2) \qquad \mathcal{J}_3(0,0;0) = O(c^2 c'^2)$$

$$\mathcal{J}_1(0,0;1) = \frac{\sqrt{3}}{6} + O(c^2, c'^2) \qquad \mathcal{J}_3(0,0;1) = \frac{\sqrt{3}}{9} c^2 + O(c^2 c'^2, c^4, c'^4)$$

$$\mathcal{J}_1(1,0;1) = \frac{3}{8} + O(c^2, c'^2) \qquad \mathcal{J}_3(1,0;1) = -\frac{3}{4} - \frac{3}{2} \ln \xi_0 + O(c^2, c'^2)$$

$$\mathcal{J}_1(1,1;1) = \frac{3}{16} + O(c^2, c'^2) \qquad \mathcal{J}_2(1,1;1) = \frac{3}{8} + O(c^2, c'^2)$$

$$\mathcal{J}_3(1,1;1) = \frac{3}{8} + O(c^2, c'^2).$$

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